A TRANSCENDENTAL APPROACH TO KOLLÁR'S INJECTIVITY THEOREM II

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Abstract. We treat a relative version of the main theorem in [F2]: A transcendental approach to Kollár's injectivity theorem. More explicitly, we give a curvature condition that implies Kollár type cohomology injectivity theorems in the relative setting. To carry out this generalization, we use Ohsawa-Takegoshi's twisted version of Nakano's identity.

Contents

1. Introduction

The following theorem is the main theorem of this paper, which is a relative version of the main theorem in [F2].

Theorem 1.1 (Main Theorem)**.** *Let $f : X \to Y$ be a proper surjective morphism from a Kähler manifold X to a complex variety Y. Let (E, h_E) (resp. (L, h_L)) be a holomorphic vector (resp. line) bundle on X with a smooth hermitian metric h_E (resp. h_L). Let F be a holomorphic line bundle on X with a singular hermitian metric h_F. Assume the following conditions.*

(i) *There exists a subvariety Z of X such that h_F is smooth on $X \setminus Z$.*

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(ii) $\sqrt{-1}\Theta(F) \geq -\widetilde{\gamma}$ *in the sense of currents, where* $\widetilde{\gamma}$ *is a smooth* $(1,1)$*-form on* X.

(iii) $\sqrt{-1}(\Theta(E) + \mathrm{Id}_E \otimes \Theta(F)) \geq_{\mathrm{Nak}} 0$ *on* $X \setminus Z$.

(iv) $\sqrt{-1}(\Theta(E) + \mathrm{Id}_E \otimes \Theta(F) - \varepsilon_0 \mathrm{Id}_E \otimes \Theta(L)) \geq_{\mathrm{Nak}} 0$ *on* $X \setminus Z$ *for some positive constant* ε_0.

Here, $\geq_{\mathrm{Nak}} 0$ *means the Nakano semi-positivity. Let* s *be a nonzero holomorphic section of* L. *Then the multiplication homomorphism*

$$\times s : R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{I}(h_F)) \longrightarrow R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{I}(h_F) \otimes L)$$

is injective for any $q \geq 0$, *where* K_X *is the canonical line bundle of* X *and* $\mathcal{I}(h_F)$ *is the multiplier ideal sheaf associated to the singular hermitian metric* h_F *of* F. *Note that* $\times s$ *is the sheaf homomorphism induced by the tensor product with* s.

For the absolute case and the background of Kollár type cohomology injectivity theorems, see the introduction of [F2]. The reader who reads Japanese may find [F3] also useful. The essential part of Theorem 1.1 is contained in Ohsawa's injectivity theorem (see [O] and [F1]). Our formulation is much more suitable for geometric applications (cf. [F2, 4. Applications]) than Ohsawa's. We note that the main ingredient of our proof is Ohsawa-Takegoshi's twisted version of Nakano's identity (cf. Proposition 2.15). The next corollary directly follows from Theorem 1.1. It contains a generalization of the Grauert-Riemenschneider vanishing theorem.

Corollary 1.2 (Torsion-freeness)**.** *Let* $f : X \to Y$ *be a proper surjective morphism from a Kähler manifold* X *to a complex variety* Y. *Let* (E, h_E) (*resp.* (F, h_F)) *be a holomorphic vector* (*resp. line*) *bundle on* X *with a smooth hermitian metric* h_E (*resp. a singular hermitian metric* h_F). *Assume the conditions* (i), (ii), *and* (iii) *in* Theorem 1.1. *Then,* $R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{I}(h_F))$ *is torsion-free for any* $q \geq 0$. *In particular,* $R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{I}(h_F)) = 0$ *for* $q > \dim X - \dim Y$.

I will describe the proof of Theorem 1.1 in Section 3, which may help the reader to understand [O] (see also [F1]).

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2. Preliminaries

In this section, we collect basic definitions and results in algebraic and analytic geometry.

2.1 (Singular hermitian metric)**.** Let L be a holomorphic line bundle on a complex manifold X.

Definition 2.2 (Singular hermitian metric)**.** A *singular hermitian metric* on L is a metric which is given in any trivialization $\theta : L|_\Omega \simeq \Omega \times \mathbb{C}$ by

$$\|\xi\| = |\theta(\xi)|e^{-\varphi(x)}, \quad x \in \Omega,\ \xi \in L_x,$$

where $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ is an arbitrary function, called the *weight* of the metric with respect to the trivialization θ. Here, $L^1_{\mathrm{loc}}(\Omega)$ is the space of the locally integrable functions on Ω.

2.3 (Multiplier ideal sheaf)**.** The notion of multiplier ideal sheaves introduced by Nadel is very important. First, we recall the notion of (quasi-)plurisubharmonic functions.

Definition 2.4 (Plurisubharmonic function)**.** Let X be a complex manifold. A function $\varphi : X \to [-\infty, \infty)$ is said to be *plurisubharmonic* (psh, for short) if, on each connected component of X,

1. φ is upper semi-continuous, and
2. φ is locally integrable and $\sqrt{-1}\partial\bar{\partial}\varphi$ is positive semi-definite as a $(1,1)$-current,

or $\varphi \equiv -\infty$. A *smooth strictly plurisubharmonic function* ψ on X is a smooth function on X such that $\sqrt{-1}\partial\bar{\partial}\psi$ is a positive definite smooth $(1,1)$-form.

Definition 2.5. A *quasi-plurisubharmonic* (quasi-psh, for short) function is a function φ which is locally equal to the sum of a psh function and of a smooth function.

Next, we define multiplier ideal sheaves.

Definition 2.6 (Multiplier ideal sheaf)**.** If φ is a quasi-psh function on a complex manifold X, the *multiplier ideal sheaf* $\mathcal{I}(\varphi) \subset \mathcal{O}_X$ is defined by

$$\Gamma(U, \mathcal{I}(\varphi)) = \{f \in \mathcal{O}_X(U);\ |f|^2 e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)\}$$

for every open set $U \subset X$. Then it is known that $\mathcal{I}(\varphi)$ is a coherent ideal sheaf of $\mathcal{O}_X$. See, for example, [D, (5.7) Proposition].

Finally, we note the definition of $\mathcal{I}(h_F)$ in Theorem 1.1.

Remark 2.7. By the assumption (ii) in Theorem 1.1, the weight of the singular hermitian metric h_F is a quasi-psh function on any trivialization. So, we can define multiplier ideal sheaves locally and check that they are independent of trivializations. Thus, we can define the multiplier ideal sheaf globally and denote it by $\mathcal{I}(h_F)$, which is an abuse of notation. It is a coherent ideal sheaf on X.

2.8 (Kähler geometry)**.** We collects the basic notion and results of hermitian and Kähler geometries (see also [D]).

Definition 2.9 (Chern connection and its curvature form)**.** Let X be a complex hermitian manifold and (E,h) a holomorphic hermitian vector bundle on X. Then there exists the *Chern connection* $D = D_{(E,h)}$, which can be split in a unique way as a sum of a $(1,0)$ and of a $(0,1)$-connection, $D = D'_{(E,h)} + D''_{(E,h)}$. By the definition of the Chern connection, $D'' = D''_{(E,h)} = \bar{\partial}$. We obtain the *curvature form* $\Theta(E) = \Theta_{(E,h)} = \Theta_h := D^2_{(E,h)}$. The subscripts might be suppressed if there is no danger of confusion.

Definition 2.10 (Inner product)**.** Let X be an n-dimensional complex manifold with the hermitian metric g. We denote by ω the *fundamental form* of g. Let (E,h) be a hermitian vector bundle on X, and u, v are E-valued (p,q)-forms with measurable coefficients, we set

$$\|u\|^2 = \int_X |u|^2 dV_\omega, \ \langle\langle u, v\rangle\rangle = \int_X \langle u, v\rangle dV_\omega,$$

where $|u|$ is the pointwise norm induced by g and h on $\Lambda^{p,q}T^*_X \otimes E$, and $dV_\omega = \frac{1}{n!}\omega^n$. More explicitly, $\langle u, v\rangle dV_\omega = {}^t u \wedge H\overline{*v}$, where ${}^t u$ is the transposed matrix of u, $*$ is the *Hodge star operator* relative to ω, and H is the (local) matrix representation of h. When we need to emphasize the metrics, we write $|u|_{g,h}$, and so on.

Let $L^{p,q}_{(2)}(X,E)(= L^{p,q}_{(2)}(X,(E,h)))$ be the space of square integrable E-valued (p,q)-forms on X. The inner product was defined in Definition 2.10. When we emphasize the metrics, we write $L^{p,q}_{(2)}(X,E)_{g,h}$, where g (resp. h) is the hermitian metric of X (resp. E). As usual one can view D' and D'' as closed and densely defined operators on the Hilbert space $L^{p,q}_{(2)}(X,E)$. The formal adjoints D'^*, D''^* also have closed extensions in the sense of distributions, which do not necessarily coincide with the Hilbert space adjoints in the sense of Von Neumann, since the latter ones may have strictly smaller domains. It is well known, however, that the domains coincide if the hermitian metric of X is complete. See Lemma 2.11 below.

Lemma 2.11 (Density Lemma)**.** *Let X be a complex manifold with the complete hermitian metric g and (E,h) a holomorphic hermitian vector bundle on X. Then $C^{p,q}_0(X,E)$ is dense in* $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(D''^*_{(E,h)})$ *with respect to the graph norm $\|v\| + \|\bar{\partial}v\| + \|D''^*_{(E,h)}v\|$, where* $\mathrm{Dom}(\bar{\partial})$ *(resp.* $\mathrm{Dom}(D''^*_{(E,h)})$*) is the domain of $\bar{\partial}$ (resp. $D''^*_{(E,h)}$).*

Suppose that (E, h) is a holomorphic hermitian vector bundle and that (e_λ) is a holomorphic frame for E over some open set U. Then the metric h is given by the $r \times r$ hermitian matrix $H = (h_{\lambda\mu})$, where $h_{\lambda\mu} = h(e_\lambda, e_\mu)$ and $r = \mathrm{rank} E$. Then we have $h(u, v) = {}^t u H \bar{v}$ on U for smooth sections u, v of $E|_U$. This implies that $h(u, v) = \sum_{\lambda,\mu} u_\lambda h_{\lambda\mu} \bar{v}_\mu$ for $u = \sum e_i u_i$ and $v = \sum e_j v_j$. Then we obtain that $\sqrt{-1}\Theta_h(E) = \sqrt{-1}\bar{\partial}(\overline{H}^{-1}\partial\overline{H})$ and ${}^t\overline{(\sqrt{-1}{}^t\Theta_h(E)H)} = \sqrt{-1}{}^t\Theta_h(E)H$ on U. Let $C^{p,q}(X, E)$ (resp. $C_0^{p,q}(X, E)$) be the space of smooth E-valued (p, q)-forms (resp. smooth E-valued (p, q)-forms with compact supports) on X. We define $\{u, v\} = {}^t u \wedge H\bar{v}$ for $u \in C^{p,q}(X, E)$ and $v \in C^{r,s}(X, E)$, where ${}^t u$ is the transposed matrix of u. We will use $\{\cdot, \cdot\}$ in Section 3.

Definition 2.12 (Nakano positivity and semi-positivity)**.** Let (E, h) be a holomorphic vector bundle on a complex manifold X with a smooth hermitian metric h. Let Ξ be a $\mathrm{Hom}(E, E)$-valued $(1, 1)$-form such that ${}^t\overline{({}^t\Xi h)} = {}^t\Xi h$. Then Ξ is said to be *Nakano positive* (resp. *Nakano semi-positive*) if the hermitian form on $T_X \otimes E$ associated to ${}^t\Xi h$ is positive definite (resp. semi-definite). We write $\Xi >_{\mathrm{Nak}} 0$ (resp. $\geq_{\mathrm{Nak}} 0$). We note that $\Xi_1 >_{\mathrm{Nak}} \Xi_2$ (resp. $\Xi_1 \geq_{\mathrm{Nak}} \Xi_2$) means that $\Xi_1 - \Xi_2 >_{\mathrm{Nak}} 0$ (resp. $\geq_{\mathrm{Nak}} 0$). A holomorphic vector bundle (E, h) is said to be *Nakano positive* (resp. *semi-positive*) if $\sqrt{-1}\Theta(E) >_{\mathrm{Nak}} 0$ (resp. $\geq_{\mathrm{Nak}} 0$). We usually omit "Nakano" when E is a line bundle.

The space of harmonic forms will play important roles in the proof of Theorem 1.1. See also the introduction of [F2].

Definition 2.13 (Harmonic forms)**.** Let X be an n-dimensional complete Kähler manifold with a complete Kähler metric g. Let (E, h_E) be a holomorphic hermitian vector bundle on X. We put

$$\mathcal{H}^{p,q}(X, (E, h_E))_g = \{u \in L^{p,q}_{(2)}(X, E) | \bar{\partial}u = 0 \text{ and } D''^{*}_{(E,h_E)}u = 0\}.$$

Note that $\mathcal{H}^{p,q}(X, (E, h_E))_g \subset C^{p,q}(X, E)$ by the regularization theorem for elliptic partial differential equations of second order.

2.14 (Ohsawa-Takegoshi twist)**.** The following formula is a *twisted* version of Nakano's identity, which is now well known to the experts.

Proposition 2.15 (Ohsawa-Takegoshi twist)**.** *Let (E, h) be a holomorphic hermitian vector bundle on an n-dimensional Kähler manifold X. Let η be any smooth positive function on X. Then, for every $u \in C_0^{n,q}(X, E)$, the equality*

$$(\spadesuit) \qquad \begin{aligned} &\|\sqrt{\eta}D''^{*}_{(E,h)}u\|^2 + \|\sqrt{\eta}\bar{\partial}u\|^2 - \|\sqrt{\eta}D'^{*}u\|^2 \\ =&\langle\!\langle \sqrt{-1}(\eta\Theta_h - \mathrm{Id}_E \otimes \partial\bar{\partial}\eta)\Lambda u, u\rangle\!\rangle + 2\mathrm{Re}\langle\!\langle \bar{\partial}\eta \wedge D''^{*}_{(E,h)}u, u\rangle\!\rangle \end{aligned}$$

holds true. Here, we denote by Λ the adjoint operator of $\omega\wedge\cdot$. Note that $D''=D''_{(E,h)}=\bar\partial$ and D'^* are independent of the hermitian metric h.

Sketch of the proof. We quickly review the proof of this proposition for the reader's convenience. If A, B are the endomorphisms of pure degree of the graded module $C^{\bullet,\bullet}(X,E)$, their *graded Lie bracket* is defined by

$$[A,B]=AB-(-1)^{\deg A\deg B}BA.$$

Let

$$\Delta'=D'D'^*+D'^*D'$$

and

$$\Delta''=D''D''^*+D''^*D''$$

be the complex Laplace operators. Then it is well known that

$$\Delta''=\Delta'+[\sqrt{-1}\Theta(E),\Lambda],$$

which is sometimes called Nakano's identity. Let us consider the *twisted* Laplace operators

$$D'\eta D'^*+D'^*\eta D'=\eta\Delta'+(\partial\eta)D'^*-(\partial\eta)^*D',$$

and

$$D''\eta D''^*+D''^*\eta D''=\eta\Delta''+(\bar\partial\eta)D''^*-(\bar\partial\eta)^*D''.$$

On the other hand, we can easily check that

$$[\sqrt{-1}\partial\bar\partial\eta,\Lambda]=[D'',(\bar\partial\eta)^*]+[D'^*,\partial\eta].$$

Combining these equalities, we find

$$\begin{aligned}&D''\eta D''^*+D''^*\eta D''-D'\eta D'^*-D'^*\eta D'+[\sqrt{-1}\partial\bar\partial\eta,\Lambda]\\ &=\eta[\sqrt{-1}\Theta(E),\Lambda]+(\bar\partial\eta)D''^*+D''(\bar\partial\eta)^*+(\partial\eta)^*D'+D'^*(\partial\eta).\end{aligned}$$

Apply this identity to a form $u\in C^{n,q}_0(X,E)$ and take the inner product with u. Then we obtain the desired formula. □

The next proposition is [O, Lemma 2.1]. The proof is a routine work. It easily follows from Lemmas 2.11 and 2.17.

Proposition 2.16. *Fix a complete Kähler metric g on X. We put*

$$D^{n,q}=\{u\in L^{n,q}_{(2)}(X,E)\mid \bar\partial u\in L^{n,q+1}_{(2)}(X,E) \text{ and } D''^*u\in L^{n,q-1}_{(2)}(X,E)\},$$

that is, $D^{n,q}=\mathrm{Dom}(\bar\partial)\cap\mathrm{Dom}(D''^*_{(E,h)})\subset L^{n,q}_{(2)}(X,E)$. Suppose that η is bounded and that there exists a constant $\varepsilon>0$ such that

$$\sqrt{-1}(\eta\Theta_h-\mathrm{Id}_E\otimes\partial\bar\partial\eta-\varepsilon\mathrm{Id}_E\otimes\partial\eta\wedge\bar\partial\eta)\geq_{\mathrm{Nak}}0$$

holds everywhere. Then the equality (♠) holds for all $u\in D^{n,q}$.

Lemma 2.17. *For any* $u \in C^{n,q}(X,E)$ *and any positive real number* δ*, we have*

$$\begin{aligned} 2\mathrm{Re}\langle\!\langle \bar\partial\eta \wedge D''^{*}_{(E,h)}u, u\rangle\!\rangle &= 2\mathrm{Re}\langle\!\langle D''^{*}_{(E,h)}u, (\bar\partial\eta)^* u\rangle\!\rangle \\ &\leq \frac{1}{\delta}\|D''^{*}_{(E,h)}u\|^2 + \delta\|(\bar\partial\eta)^*u\|^2, \text{ and} \end{aligned}$$

$$\begin{aligned} \|(\bar\partial\eta)^*u\|^2 &= \langle\!\langle (\bar\partial\eta)^*u, (\bar\partial\eta)^*u\rangle\!\rangle \\ &= \langle\!\langle \sqrt{-1}\partial\eta\wedge\bar\partial\eta\Lambda u, u\rangle\!\rangle \end{aligned}$$

since $(\bar\partial\eta)^*u = -\sqrt{-1}\partial\eta\Lambda u$ *for* $u \in C^{n,q}(X,E)$*. Note that* $(\bar\partial\eta)^*$ *is the adjoint operator of* $\bar\partial\eta\wedge\cdot$ *relative to the inner product* $\langle\ ,\ \rangle$.

We close this section by the following remark on [T].

Remark 2.18. By Proposition 2.16, we can prove [T, Theorem 3.4 (ii)] under the slightly weaker assumption that φ is a *bounded* smooth psh function on M. We do not have to assume that $|d\varphi|$ is bounded on M. For the notations, see [T]. In this case, there are positive constants C_1 and C_2 such that $\varphi + C_1 > 0$ on M and $C_2 - (\varphi + C_1)^2 > 0$ on M. We can use Proposition 2.16 (and Lemma 2.17) for $\eta := C_2 - (\varphi + C_1)^2$ and $\varepsilon := \frac{1}{2C_2} > 0$. Then we obtain $(\bar\partial\varphi)^*u = 0$ and $\langle\sqrt{-1}\partial\bar\partial\varphi\Lambda u, u\rangle_h = 0$.

3. Proof of the main theorem

In this section, we prove Theorem 1.1. So, we freely use the notation in Theorem 1.1. Let $W \Subset Y$ be any Stein open subset. We put $V = f^{-1}(W)$. Then V is a holomorphically convex weakly 1-complete Kähler manifold. To prove Theorem 1.1, it is sufficient to show that

$$\times s : H^q(V, K_V \otimes E \otimes F \otimes \mathcal{I}(h_F)) \longrightarrow H^q(V, K_V \otimes E \otimes F \otimes \mathcal{I}(h_F) \otimes L)$$

is injective for any $q \geq 0$. Note that the above cohomology groups are separated topological vector spaces since V is holomorphically convex.

Remark 3.1. A weakly 1-complete manifold X is called a *weakly pseudoconvex manifold* in [D]. A weakly 1-complete manifold is a complex manifold equipped with a *smooth* plurisubharmonic exhaustion function. More explicitly, there exists a smooth plurisubharmonic function φ on X such that $X_c = \{x \in X | \varphi(x) < c\}$ is relatively compact in X for any c.

We define bounded smooth functions from the given nonzero holomorphic section s of L.

Definition 3.2. Take a smooth plurisubharmonic exhaustion function φ on V. Without loss of generality, we can assume that $\min_{x\in V}\varphi(x)=0$. Let s be a holomorphic section of L. Let $|s|$ be the pointwise norm of s with respect to the fiber metric h_L. Let $\lambda : [0,\infty) \longrightarrow [0,\infty)$ be a smooth convex increasing function such that $4|s|^2 < e^{\lambda(\varphi)}$. Thus, $|s|^2_{\lambda(\varphi)} < \frac{1}{4}$, where $|s|_{\lambda(\varphi)}$ is the pointwise norm of s with respect to the fiber metric $h_L e^{-\lambda(\varphi)}$. We put $\mu(x) := \lambda(x) + x$. Obviously, μ is also a smooth convex increasing function.

Definition 3.3. We put $\chi(t) = t - \log(-t)$ for $t < 0$. We define

$$\begin{aligned}\sigma_{\varepsilon,\lambda} &= \log(|s|^2_{\lambda(\varphi)} + \varepsilon), \text{ and}\\ \eta_{\varepsilon,\lambda} &= \frac{1}{\epsilon} - \chi(\sigma_{\varepsilon,\lambda})\\ &= -\log(|s|^2_{\lambda(\varphi)} + \varepsilon) + \log(-\log(|s|^2_{\lambda(\varphi)} + \varepsilon)) + \frac{1}{\varepsilon}.\end{aligned}$$

We can also define $\sigma_{\varepsilon,\mu}$ and $\eta_{\varepsilon,\mu}$ similarly. Note that $\eta_{\varepsilon,\lambda}$ and $\eta_{\varepsilon,\mu}$ are smooth bounded functions on V with $\eta_{\varepsilon,\mu} \geq \eta_{\varepsilon,\lambda} > \frac{1}{\varepsilon}$. The subscripts λ, μ, and ε might be suppressed if there is no danger of confusion.

We note the following obvious remark before we go to various calculations.

Remark 3.4. We note that $e < 2\sqrt{2}$. Thus, $\frac{3}{2}\log 2 > 1$. Therefore, $\sigma_{\varepsilon,\mu} \leq \sigma_{\varepsilon,\lambda} < 2\log\frac{1}{2} < -\frac{4}{3}$ if ε is small since $|s|^2_{\mu(\varphi)} \leq |s|^2_{\lambda(\varphi)} < \frac{1}{4}$. Of course, $\log(-\sigma_{\varepsilon,\mu}) \geq \log(-\sigma_{\varepsilon,\lambda}) > \log\frac{4}{3} > 0$. We have $\chi'(t) = 1 - \frac{1}{t}$ and $\chi''(t) = \frac{1}{t^2}$. Thus, $1 < \chi'(\sigma_{\varepsilon,\mu}) \leq \chi'(\sigma_{\varepsilon,\lambda}) = 1 + \frac{1}{(-\sigma_{\varepsilon,\lambda})} < \frac{7}{4}$.

3.5 (Basic calculations)**.** We calculate various differentials of $\eta_{\varepsilon,\lambda}$. The same arguments work for $\eta_{\varepsilon,\mu}$.

Definition 3.6. Let $u \in C^{p,q}(V,L)$ and $v \in C^{r.s}(V,L)$. We define

$$\{u,v\}_{\lambda(\varphi)} = u \wedge H_L e^{-\lambda(\varphi)}\bar{v},$$

where H_L is the local matrix representation of h_L.

We have

$$\partial\sigma_{\varepsilon,\lambda} = \frac{\{D's, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)} + \varepsilon}.$$

In the above equation, D' is the $(1,0)$ part of the Chern connection of $L' = (L, h_L e^{-\lambda(\varphi)})$, that is, $D' = D'_{(L,h_L e^{-\lambda(\varphi)})}$. Thus, $\Theta(L') = \Theta(L) +$

$\partial\bar{\partial}\lambda(\varphi)$. We obtain the following equation by the direct computation.

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\sigma_{\varepsilon,\lambda} = &-\frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)}+\varepsilon} \\ &+\frac{\sqrt{-1}\{D's, D's\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)}+\varepsilon} - \frac{\sqrt{-1}\{D's, s\}_{\lambda(\varphi)} \wedge \{s, D's\}_{\lambda(\varphi)}}{(|s|^2_{\lambda(\varphi)}+\varepsilon)^2},\end{aligned}$$

where $L' = (L, h_L e^{-\lambda(\varphi)})$. By the Cauchy-Schwartz inequality, we have

$$\sqrt{-1}\{D's, D's\}_{\lambda(\varphi)}|s|^2_{\lambda(\varphi)} \geq \sqrt{-1}\{D's, s\}_{\lambda(\varphi)} \wedge \{s, D's\}_{\lambda(\varphi)}.$$

Substituting the Cauchy-Schwartz inequality into the above equation, we obtain

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\sigma_{\varepsilon,\lambda} &\geq \frac{\varepsilon}{|s|^2_{\lambda(\varphi)}(|s|^2_{\lambda(\varphi)}+\varepsilon)^2}\sqrt{-1}\{D's, s\}_{\lambda(\varphi)} \wedge \{s, D's\}_{\lambda(\varphi)} \\ &\quad - \frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)}+\varepsilon} \\ &= \frac{\varepsilon}{|s|^2_{\lambda(\varphi)}}\sqrt{-1}\partial\sigma_{\varepsilon,\lambda} \wedge \bar{\partial}\sigma_{\varepsilon,\lambda} - \frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)}+\varepsilon}.\end{aligned}$$

Lemma 3.7. *The next equations follow easily from the definition.*

$$\begin{aligned}\partial\eta_{\varepsilon,\lambda} &= -\chi'(\sigma_{\varepsilon,\lambda})\partial\sigma_{\varepsilon,\lambda}, \\ \bar{\partial}\eta_{\varepsilon,\lambda} &= -\chi'(\sigma_{\varepsilon,\lambda})\bar{\partial}\sigma_{\varepsilon,\lambda}, \\ \partial\bar{\partial}\eta_{\varepsilon,\lambda} &= -\chi''(\sigma_{\varepsilon,\lambda})\partial\sigma_{\varepsilon,\lambda} \wedge \bar{\partial}\sigma_{\varepsilon,\lambda} - \chi'(\sigma_{\varepsilon,\lambda})\partial\bar{\partial}\sigma_{\varepsilon,\lambda}.\end{aligned}$$

Combining the above (in)equalities, we have

$$\begin{aligned}-\sqrt{-1}\partial\bar{\partial}\eta_{\varepsilon,\lambda} &= \chi'(\sigma_{\varepsilon,\lambda})\sqrt{-1}\partial\bar{\partial}\sigma_{\varepsilon,\lambda} + \chi''(\sigma_{\varepsilon,\lambda})\sqrt{-1}\partial\sigma_{\varepsilon,\lambda} \wedge \bar{\partial}\sigma_{\varepsilon,\lambda} \\ &\geq \frac{\varepsilon\chi'(\sigma_{\varepsilon,\lambda})}{|s|^2_{\lambda(\varphi)}}\sqrt{-1}\partial\sigma_{\varepsilon,\lambda} \wedge \bar{\partial}\sigma_{\varepsilon,\lambda} + \chi''(\sigma_{\varepsilon,\lambda})\sqrt{-1}\partial\sigma_{\varepsilon,\lambda} \wedge \bar{\partial}\sigma_{\varepsilon,\lambda} \\ &\quad - \chi'(\sigma_{\varepsilon,\lambda})\frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)}+\varepsilon} \\ &= \left(\frac{\varepsilon}{\chi'(\sigma_{\varepsilon,\lambda})|s|^2_{\lambda(\varphi)}} + \frac{\chi''(\sigma_{\varepsilon,\lambda})}{\chi'(\sigma_{\varepsilon,\lambda})^2}\right)\sqrt{-1}\partial\eta_{\varepsilon,\lambda} \wedge \bar{\partial}\eta_{\varepsilon,\lambda} \\ &\quad - \chi'(\sigma_{\varepsilon,\lambda})\frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|^2_{\lambda(\varphi)}+\varepsilon}.\end{aligned}$$

Lemma 3.8. *We have the following inequality.*

$$\frac{\chi''(\sigma_{\varepsilon,\lambda})}{\chi'(\sigma_{\varepsilon,\lambda})^2} \geq \eta_{\varepsilon,\lambda}^{-2}$$

Proof. By the definition, it is easy to see that

$$\frac{\chi'(\sigma_{\varepsilon,\lambda})}{\chi''(\sigma_{\varepsilon,\lambda})}=\frac{(1-\frac{1}{\sigma_{\varepsilon,\lambda}})^2}{\frac{1}{\sigma^2_{\varepsilon,\lambda}}}=(\sigma_{\varepsilon,\lambda}-1)^2.$$

On the other hand, $\eta_{\varepsilon,\lambda}=\frac{1}{\varepsilon}-\sigma_{\varepsilon,\lambda}+\log(-\sigma_{\varepsilon,\lambda})>1-\sigma_{\varepsilon,\lambda}$. Thus, we obtain the desired inequality. □

Proposition 3.9. *On the curvature conditions*

$$\sqrt{-1}(\Theta(E)+\mathrm{Id}_E\otimes\Theta(F))\geq_{\mathrm{Nak}}0$$

on $X\setminus Z$ and

$$\sqrt{-1}(\Theta(E)+\mathrm{Id}_E\otimes\Theta(F)-\varepsilon_0\mathrm{Id}_E\otimes\Theta(L))\geq_{\mathrm{Nak}}0$$

on $X\setminus Z$ for some positive real number ε_0, we have a small positive real number ε_1 such that

$$\sqrt{-1}(\eta\Theta_{(E\otimes F,h_Eh_Fe^{-\mu(\varphi)})}-\mathrm{Id}_E\otimes\partial\bar{\partial}\eta)\geq_{\mathrm{Nak}}\sqrt{-1}(\mathrm{Id}_E\otimes\eta^{-2}\partial\eta\wedge\bar{\partial}\eta)$$

holds on $V\setminus Z$ for $0<\varepsilon<\varepsilon_1$, where $\eta=\eta_{\varepsilon,\lambda}$ or $\eta_{\varepsilon,\mu}$.

Proof. By the definitions of λ and μ, $\partial\bar{\partial}\mu(\varphi)=\partial\bar{\partial}\lambda(\varphi)+\partial\bar{\partial}\varphi$, and $\lambda(\varphi)$ and $\mu(\varphi)$ are plurisubharmonic. Note that

$$\begin{aligned}\Theta_{(E\otimes F,h_Eh_Fe^{-\mu(\varphi)})}&=\Theta(E)+\mathrm{Id}_E\otimes\Theta(F)+\mathrm{Id}_E\otimes\partial\bar{\partial}\mu(\varphi)\\&=\Theta(E)+\mathrm{Id}_E\otimes\Theta(F)+\mathrm{Id}_E\otimes\partial\bar{\partial}\lambda(\varphi)+\mathrm{Id}_E\otimes\partial\bar{\partial}\varphi,\\\Theta_{(L,h_Le^{-\mu(\varphi)})}&=\Theta(L)+\partial\bar{\partial}\mu(\varphi),\ \text{and}\\\Theta_{(L,h_Le^{-\lambda(\varphi)})}&=\Theta(L)+\partial\bar{\partial}\lambda(\varphi).\end{aligned}$$

Therefore, the desired inequality follows from the calculations in 3.5. □

In the next lemma, we obtain the relationship between the Chern connections of (L,h_L) and $(L,h_Le^{-\lambda(\varphi)})$.

Lemma 3.10. *Let $\gamma:[0,\infty)\longrightarrow\mathbb{R}$ be any smooth $\mathbb{R}$-valued function. Then we have the following equation by the definition of the Chern connection.*

$$\begin{aligned}D'_{(L,h_Le^{-\gamma(\varphi)})}&=(H_Le^{-\gamma(\varphi)})^{-1}\partial(H_Le^{-\gamma(\varphi)}\cdot\ )\\&=\partial+\partial\log(H_Le^{-\gamma(\varphi)})\wedge\cdot\\&=\partial+\partial\log H_L\wedge\cdot\ -\gamma'(\varphi)\partial\varphi\wedge\cdot\\&=D'_{(L,h_L)}-\gamma'(\varphi)\partial\varphi\wedge\cdot\ .\end{aligned}$$

We note that $H_L=\overline{H}_L$ since L is a line bundle.

3.11 (Complete Kähler metrics)**.** There exists a complete Kähler metric g on V since V is weakly 1-complete. Let ω be the fundamental form of g. We note the following well-known lemma.

Lemma 3.12. *There exists a quasi-psh function ψ on X such that $\psi = -\infty$ on Z with logarithmic poles along Z and ψ is smooth outside Z.*

Without loss of generality, we can assume that $\psi \le -e$ on $V \Subset X$. We put $\widetilde{\psi} = \frac{1}{\log(-\psi)}$. Then $\widetilde{\psi}$ is a quasi-psh function on V and $\widetilde{\psi} \le 1$. Thus, we can take a positive constant α such that $\sqrt{-1}\partial\bar{\partial}\widetilde{\psi} + \alpha\omega > 0$ on $V \setminus Z$. Let g' be the Kähler metric on $V \setminus Z$ whose fundamental form is $\omega' = \omega + (\sqrt{-1}\partial\bar{\partial}\widetilde{\psi} + \alpha\omega)$. We note that we can check that

$$\omega' \ge \sqrt{-1}\partial(\log(\log(-\psi))) \wedge \bar{\partial}(\log(\log(-\psi)))$$

if we choose $\alpha \gg 0$. It is because

$$\partial\bar{\partial}\widetilde{\psi} = 2\frac{\frac{-\partial\psi}{-\psi} \wedge \frac{-\bar{\partial}\psi}{-\psi}}{(\log(-\psi))^3} + \frac{\frac{\partial\bar{\partial}\psi}{-\psi}}{(\log(-\psi))^2} + \frac{\frac{-\partial\psi\wedge(-\bar{\partial}\psi)}{(-\psi)^2}}{(\log(-\psi))^2},$$

and

$$\partial(\log(\log(-\psi))) = \frac{\frac{-\partial\psi}{-\psi}}{\log(-\psi)}.$$

Therefore, g' is a *complete* Kähler metric on $V \setminus Z$ by Hopf-Rinow because $\log(\log(-\psi))$ tends to $+\infty$ on Z. We fix these Kähler metrics throughout this proof.

3.13 (Key Results)**.** The following three propositions are the heart of the proof of Theorem 1.1.

Proposition 3.14. *For any $u \in \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$, $(\bar{\partial}\eta)^* u = 0$ for $\eta = \eta_{\varepsilon,\lambda}$ and $\eta_{\varepsilon,\mu}$. This implies that $\partial\eta \wedge *u = 0$ for $\eta = \eta_{\varepsilon,\lambda}$ and $\eta_{\varepsilon,\mu}$. Thus, we obtain $D'_{(L,h_L e^{-\lambda(\varphi)})} s \wedge *u = 0$ and $D'_{(L,h_L e^{-\mu(\varphi)})} s \wedge *u = 0$.*

Proof. The definition of $\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$ implies that $\bar{\partial}u = 0$ and $D''^{*}_{(E\otimes F, h_E h_F e^{-\mu(\varphi)})} u = 0$. By Propositions 2.15, 2.16, and 3.9, we have

$$0 \ge -\|\sqrt{\eta} D'^{*} u\|^2 \ge \langle\!\langle \sqrt{-1}\eta^{-2}\partial\eta \wedge \bar{\partial}\eta \Lambda u, u \rangle\!\rangle \ge 0.$$

Thus, we have $(\bar{\partial}\eta)^* u = 0$ (cf. Lemma 2.17). It is not difficult to see that $(\bar{\partial}\eta)^* u = 0$ implies $\partial\eta \wedge *u = 0$. By the definition of η, we obtain $D'_{(L,h_L e^{-\lambda(\varphi)})} s \wedge *u = 0$ (resp. $D'_{(L,h_L e^{-\mu(\varphi)})} s \wedge *u = 0$) if $\eta = \eta_{\varepsilon,\lambda}$ (resp. $\eta = \eta_{\varepsilon,\mu}$). □

Proposition 3.15. *If $D'_{(L,h_Le^{-\lambda(\varphi)})}s\wedge *u=0$ and $D'_{(L,h_Le^{-\mu(\varphi)})}s\wedge *u=0$, then $D'_{(L,h_L)}s\wedge *u=0$ and $\partial\varphi\wedge *u=0$. Therefore, $D'_{(L,h_Le^{-\nu(\varphi)})}s\wedge *u=0$ for any smooth $\mathbb{R}$-valued function ν defined on $[0,\infty)$.*

Proof. We note that $D'_{(L,h_Le^{-\lambda(\varphi)})}=D'_{(L,h_L)}-\lambda'(\varphi)\partial\varphi\wedge\cdot$ and

$$D'_{(L,h_Le^{-\mu(\varphi)})}=D'_{(L,h_L)}-\lambda'(\varphi)\partial\varphi\wedge\cdot-\partial\varphi\wedge\cdot$$

since $\mu(x)=\lambda(x)+x$. $\square$

Proposition 3.16. *If $D''^{*}_{(E\otimes F,h_Eh_Fe^{-\mu(\varphi)})}u=0$, then we obtain*

$$D''^{*}_{(E\otimes F\otimes L,h_Eh_Fh_Le^{-\mu(\varphi)-\nu(\varphi)})}(su)=0$$

for any smooth $\mathbb{R}$-valued function ν defined on $[0,\infty)$.

Proof. Let H_E (resp. H_F) be the local matrix representation of h_E (resp. h_F). The condition $D''^{*}_{(E\otimes F,h_Eh_Fe^{-\mu(\varphi)})}u=0$ implies that

$$\bar\partial(e^{-\mu(\varphi)}H_EH_F\overline{*u})=0.$$

To prove $D''^{*}_{(E\otimes F\otimes L,h_Eh_Fh_Le^{-\mu(\varphi)-\nu(\varphi)})}(su)=0$, it is sufficient to check that $\bar\partial(H_EH_Fe^{-\mu(\varphi)-\nu(\varphi)}H_L\overline{*su})=0$. We note that

$$\bar\partial(H_EH_Fe^{-\mu(\varphi)-\nu(\varphi)}H_L\overline{*su})=\bar\partial(H_L\overline{s}e^{-\nu(\varphi)})\wedge e^{-\mu(\varphi)}H_EH_F\overline{*u}$$

by the above condition. The right hand side is zero since $D'_{(L,H_Le^{-\nu(\varphi)})}s\wedge *u=0$. $\square$

The next theorem is a key result.

Theorem 3.17 (cf. [O, Proposition 3.1])**.** *For any smooth $\mathbb{R}$-valued function defined on $[0,\infty)$ such that $\nu\geq C$ for some constant C,*

$$s\mathcal{H}^{n,q}(V\setminus Z,(E\otimes F,h_Eh_Fe^{-\mu(\varphi)}))_{g'}$$

is contained in

$$\mathcal{H}^{n,q}(V\setminus Z,(E\otimes F\otimes L,h_Eh_Fh_Le^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)}))_{g'}$$

for all q.

Proof. Let $u\in\mathcal{H}^{n,q}(V\setminus Z,(E\otimes F,h_Eh_Fe^{-\mu(\varphi)}))_{g'}$. Then it is obvious that $su\in L^{n,q}_{(2)}(E\otimes F\otimes L,h_Eh_Fh_Le^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)})$ because $|s|^2_{\lambda(\varphi)}<\frac{1}{4}$ and $0<e^{-\nu(\varphi)}\leq e^{-C}$. So, the claim is a direct consequence of Proposition 3.16. Note that $\bar\partial(su)=0$ for $u\in\mathcal{H}^{n,q}(V\setminus Z,(E\otimes F,h_Eh_Fe^{-\mu(\varphi)}))_{g'}$ since s is holomorphic and $\bar\partial u=0$. $\square$

3.18 (Cohomology groups)**.** Before we go to the proof of the main theorem, we represent the cohomology groups on V by the objects on $V\setminus Z$.

Definition 3.19 (Space of locally square integrable forms)**.** We define the space of locally (in V) square integrable $E\otimes F$-valued (n,q)-forms on $V\setminus Z$. It is denoted by $L^{n,q}_{loc,V}(V\setminus Z, E\otimes F)$ or $L^{n,q}_{loc,V}(V\setminus Z,(E\otimes F, h_Eh_F))$. The vector space $L^{n,q}_{loc,V}(V\setminus Z, E\otimes F)$ is spanned by (n,q)-forms u on $V\setminus Z$ with measurable coefficients such that

$$\int_U |u|^2_{g',h_Eh_F}dV_{\omega'}<\infty$$

for any $U\Subset V$ (not $U\Subset V\setminus Z$), where $|\cdot|_{g',h_Eh_F}$ is the pointwise norm with respect to g' and h_Eh_F. We note the following obvious remark. Let $h: V\longrightarrow (0,\infty)$ be a smooth positive function. Then $L^{n,q}_{loc,V}(V\setminus Z,(E\otimes F,h_Eh_F)) = L^{n,q}_{loc,V}(V\setminus Z,(E\otimes F,h_Eh_Fh))$. We can define $L^{n,q}_{loc,V}(V\setminus Z, E\otimes F\otimes L)$ similarly.

The next lemma is essentially the same as [F2, Claim 1], which is more or less known (cf. [T, Proposition 4.6]).

Lemma 3.20. *The following isomorphism holds.*

$$\begin{aligned} & H^q(V, K_V\otimes E\otimes F\otimes \mathcal{I}(h_F)) \\ \simeq\ & H^{n,q}_{loc,V}(V\setminus Z, E\otimes F)_{g'} \\ :=\ & \frac{\operatorname{Ker}\bar\partial\cap L^{n,q}_{loc,V}(V\setminus Z, E\otimes F)}{L^{n,q}_{loc,V}(V\setminus Z, E\otimes F)\cap \bar\partial L^{n,q-1}_{loc,V}(V\setminus Z, E\otimes F)}. \end{aligned}$$

Sketch of the proof. Let $V=\bigcup_{i\in I}U_i$ be a locally finite Stein cover of V such that each U_i is sufficiently small and $U_i\Subset V$. We denote this cover by $\mathcal{U}=\{U_i\}_{i\in I}$. By Cartan and Leray, we obtain $H^q(V,K_V\otimes E\otimes F\otimes\mathcal{I}(h_F))\simeq \check{H}^q(\mathcal{U},K_V\otimes E\otimes F\otimes \mathcal{I}(h_F))$, where the right hand side is the Čech cohomology group calculated by $\mathcal{U}$. By using a partition of unity $\{\rho_i\}_{i\in I}$ associated to $\mathcal{U}$, we can construct a homomorphism

$$\rho:\check{H}^q(\mathcal{U},K_X\otimes E\otimes F\otimes\mathcal{I}(h_F))\longrightarrow H^{n,q}_{loc,V}(V\setminus Z,E\otimes F)_{g'}.$$

See Remark 3.21 below. We can check that ρ is an isomorphism. Note the following facts: (a) The open set $U_{i_0}\cap\cdots\cap U_{i_k}$ is Stein. So, $U_{i_0}\cap\cdots\cap U_{i_k}\setminus Z$ is a complete Kähler manifold. Therefore, $E\otimes F$-valued $\bar\partial$-equations can be solved with suitable L^2 estimates on $U_{i_0}\cap\cdots\cap U_{i_k}\setminus Z$ by Lemma 3.22 below. (b) Let U be an open subset of V. An $E\otimes F$-valued holomorphic $(n,0)$-form on $U\setminus Z$ with a finite L^2 norm can be extended to an $E\otimes F$-valued holomorphic $(n,0)$-form on U (cf. Remark 3.21). □

Remark 3.21. Let u be an $E\otimes F$-valued (n,q)-form on $V\setminus Z$ with measurable coefficients. Then, we have $|u|^2_{g',h_Eh_F}dV_{\omega'}\le |u|^2_{g,h_Eh_F}dV_\omega$, where $|u|_{g',h_Eh_F}$ (resp. $|u|_{g,h_Eh_F}$) is the pointwise norm induced by g'

and h_Eh_F (resp. g and h_Eh_F) since $g' > g$ on $V \setminus Z$. If u is an $E \otimes F$-valued $(n,0)$-form, then $|u|^2_{g',h_Eh_F} dV_{\omega'} = |u|^2_{g,h_Eh_F} dV_\omega$.

The following lemma is [F2, Lemma 3.2], which is a reformulation of the classical L^2-estimates for $\bar\partial$-equations for our purpose.

Lemma 3.22 (L^2-estimates for $\bar\partial$-equations on complete Kähler manifolds)**.** *Let U be a sufficiently small Stein open set of V. If $u \in L^{n,q}_{(2)}(U \setminus Z, E \otimes F)_{g',h_Eh_F}$ with $\bar\partial u = 0$, then there exists $v \in L^{n,q-1}_{(2)}(U \setminus Z, E \otimes F)_{g',h_Eh_F}$ such that $\bar\partial v = u$. Moreover, there exists a positive constant C independent of u such that*

$$\int_{U\setminus Z} |v|^2_{g',h_Eh_F} \leq C \int_{U\setminus Z} |u|^2_{g',h_Eh_F}.$$

By the same arguments, the isomorphism in Lemma 3.20 holds even when we replace $(E \otimes F, h_Eh_F)$ with $(E \otimes F \otimes L, h_Eh_Fh_L)$.

3.23 (Proof of the main theorem)**.** Let us go to the proof of Theorem 1.1 (cf. [O] and [F1, 3.15]).

Proof of Theorem 1.1. Let u be any $\bar\partial$-closed locally square integrable $E \otimes F$-valued (n,q)-form on $V \setminus Z$ such that $su = \bar\partial v$ for some $v \in L^{n,q-1}_{loc,V}(V \setminus Z, E \otimes F \otimes L)$. We choose λ such that $|s|^2_{\lambda(\varphi)} < \frac{1}{4}$ and $u \in L^{n,q}_{(2)}(V \setminus Z, (E \otimes F, h_Eh_Fe^{-\lambda(\varphi)}))$. Since $\mu(\varphi) = \lambda(\varphi) + \varphi$, $u \in L^{n,q}_{(2)}(V \setminus Z, (E \otimes F, h_Eh_Fe^{-\mu(\varphi)}))$. By choosing ν suitably, we can assume that $v \in L^{n,q-1}_{(2)}(V \setminus Z, (E \otimes F \otimes L, h_Eh_Fh_Le^{-\mu(\varphi)-\nu(\varphi)}))$. In particular, $v \in L^{n,q-1}_{(2)}(V \setminus Z, (E \otimes F \otimes L, h_Eh_Fh_Le^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)}))$. Let Pu be the orthogonal projection of u to $\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_Eh_Fe^{-\mu(\varphi)}))_{g'}$. We note that

$$\begin{aligned} & L^{n,q}_{(2)}(V \setminus Z, (E \otimes F, h_Eh_Fe^{-\mu(\varphi)})) \\ = \ & \overline{\mathrm{Im}\bar\partial} \oplus \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_Eh_Fe^{-\mu(\varphi)}))_{g'} \oplus \overline{\mathrm{Im}D''^{*}_{(E\otimes F, h_Eh_Fe^{-\mu(\varphi)})}}, \end{aligned}$$

and

$$\mathrm{Ker}\bar\partial \ = \ \overline{\mathrm{Im}\bar\partial} \oplus \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_Eh_Fe^{-\mu(\varphi)}))_{g'}.$$

Here, $\overline{\mathrm{Im}\bar\partial}$ (resp. $\overline{\mathrm{Im}D''^{*}_{(E\otimes F, h_Eh_Fe^{-\mu(\varphi)})}}$) denotes the closure of $\bar\partial C^{n,q-1}_0(V \setminus Z, E \otimes F)$ (resp. $D''^{*}_{(E\otimes F, h_Eh_Fe^{-\mu(\varphi)})} C^{n,q+1}_0(V \setminus Z, E \otimes F)$) in $L^{n,q}_{(2)}(V \setminus Z, (E \otimes F, h_Eh_Fe^{-\mu(\varphi)}))$. Note that the fixed Kähler metric g' is complete. Therefore, $u - Pu$ is in the closure of the image of $\bar\partial$. Thus, so is $s(u - Pu)$ since s is holomorphic. On the other hand,

$$sPu \in \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F \otimes L, h_Eh_Fh_Le^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)}))_{g'}$$

by Theorem 3.17. So, sPu coincides with the orthogonal projection of su to $\mathcal{H}^{n,q}(V\setminus Z, (E\otimes F\otimes L, h_Eh_Fh_Le^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)}))_{g'}$, which must be equal to zero since $v \in L^{n,q-1}_{(2)}(V\setminus Z, (E\otimes F\otimes L, h_Eh_Fh_Le^{-\mu(\varphi)-\lambda(\varphi)-\nu(\varphi)}))$. Therefore, $Pu = 0$. Since $H^q(V, K_V \otimes E \otimes F \otimes \mathcal{I}(h_F))$ is a separated topological vector space and $u \in \overline{\mathrm{Im}\bar\partial}$, there exists $w \in L^{n,q-1}_{loc,V}(V\setminus Z, E\otimes F)$ such that $u = \bar\partial w$ (cf. [T, Proposition 4.6] and [F2, Claim 1]). This means that u represents zero in $H^q(V, K_V \otimes E \otimes F \otimes \mathcal{I}(h_F))$. □

3.24 (Corollaries)**.** The proof of Corollary 1.2 is obvious if we apply Theorem 1.1 for $L = \mathcal{O}_X \simeq f^*\mathcal{O}_Y$.

Proof of Corollary 1.2*.* The statement is local. So, we can assume that Y is Stein. Let $s \in H^0(Y, \mathcal{O}_Y)$ be an arbitrary nonzero section. By Theorem 1.1,

$$\times s : R^qf_*(K_X \otimes E \otimes F \otimes \mathcal{I}(h_F)) \longrightarrow R^qf_*(K_X \otimes E \otimes F \otimes \mathcal{I}(h_F))$$

is injective for any $q \geq 0$. Thus, $R^qf_*(K_X \otimes E \otimes F \otimes \mathcal{I}(h_F))$ is torsion-free for any $q \geq 0$. □

3.25. The following proposition is a slight generalization of Theorem 1.1.

Proposition 3.26. *In* Theorem 1.1*, we can weaken the assumption that X is Kähler as follows. For any point $P \in Y$, there exist an open neighborhood U of P and a proper bimeromorphic morphism $g : W \to V := f^{-1}(U)$ from a Kähler manifold W.*

Sketch of the proof. We can find a closed subvariety Z' of W such that $W \setminus Z' \to V \setminus Z$ is an isomorphism. We can apply Corollary 1.2 to $g : W \to V$, Z', (g^*E, g^*h_E), and (g^*F, g^*h_F). Then we obtain $R^qg_*(K_W \otimes g^*E \otimes g^*F \otimes \mathcal{I}(g^*h_F)) = 0$ for any $q > 0$ and it is well known (and easy to check) that $g_*(K_W \otimes g^*E \otimes g^*F \otimes \mathcal{I}(g^*h_F)) \simeq K_V \otimes E \otimes F \otimes \mathcal{I}(h_F)$. Therefore, by Leray's spectral sequence, $R^q(f \circ g)_*(K_W \otimes g^*E \otimes g^*F \otimes \mathcal{I}(g^*h_F)) \simeq R^qf_*(K_V \otimes E \otimes F \otimes \mathcal{I}(h_F))$ for any $q \geq 0$. Apply Theorem 1.1 to $f \circ g : W \to U$, Z', (g^*E, g^*h_E), (g^*L, g^*h_L), and (g^*F, g^*h_F). Then we obtain that

$$\times s : R^q(K_X \otimes E \otimes F \otimes \mathcal{I}(h_F)) \longrightarrow R^qf_*(K_X \otimes E \otimes F \otimes \mathcal{I}(h_F) \otimes L)$$

is injective for any $q \geq 0$ by the above isomorphisms. □

The final result in this paper is related to the main theorem in [Le].

Corollary 3.27. *Let $f : X \to \Delta$ be a smooth* projective *surjective morphism from a Kähler manifold X to a disk Δ and E a Nakano semi-positive holomorphic hermitian vector bundle on X. Assume that*

there exists $D \in |K_{X_0}^{\otimes l}|$ such that $\mathcal{I}_{X_0}(cD) \simeq \mathcal{O}_{X_0}$ for any $0 \le c < 1$, where $0 \in Y$ and $X_0 = f^{-1}(0)$. Then there exists an open set $U \subset \Delta$ such that $0 \in U$ and $R^q f_*(K_X^{\otimes m} \otimes E)$ is locally free on U for any $q \ge 0$ and $1 \le m \le l$. Equivalently, $\dim_{\mathbb{C}} H^q(X_t, K_{X_t}^{\otimes m} \otimes E)$ is constant for any $q \ge 0$ by the base change theorem, where $t \in U$ and $X_t = f^{-1}(t)$.

Proof. In this proof, we shrink Δ without mentioning it for simplicity of notation. Let $s_0 \in H^0(X_0, K_{X_0}^{\otimes l})$ such that $D = (s_0 = 0)$. By Siu's extension theorem (see [S, Theorem 0.1]), there exists $s \in H^0(X, K_X^{\otimes l})$ such that $s|_{X_0} = s_0$. We consider the singular hermitian metric $h_{K_X^{\otimes(m-1)}} = (\frac{1}{|s|^2})^{\frac{m-1}{l}}$ of $K_X^{\otimes(m-1)}$. By Ohsawa-Takegoshi extension theorem and the assumption $\mathcal{I}_{X_0}(cD) \simeq \mathcal{O}_{X_0}$ for $0 \le c < 1$, we obtain that $\mathcal{I}_X(h_{K_X^{\otimes m-1}}) \simeq \mathcal{O}_X$ in a neighborhood of X_0. Therefore, we obtain that $R^q f_*(K_X^{\otimes m} \otimes E) = R^q f_*(K_X \otimes K_X^{\otimes(m-1)} \otimes E \otimes \mathcal{I}_X(h_{K_X^{\otimes m-1}}))$ is locally free by Corollary 1.2. $\square$

4. Appendix: nef, semi-positive, and semi-ample line bundles

In this appendix, we collect some examples of nef, semi-positive, and semi-ample line bundles. These examples help us understand our results in [F2] and this paper. First, we recall the following well-known example. It implies that there exists a nef line bundle that has no smooth hermitian metrics with semi-positive curvatures.

Example 4.1 (cf. [DPS, Example 1.7])**.** Let C be an elliptic curve and $\mathcal{E}$ the rank two vector bundle on C which is defined by the unique non-splitting extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

We consider the ruled surface $X := \mathbb{P}_C(\mathcal{E})$. On that surface there is a unique section $D := \mathbb{P}_C(\mathcal{O}_C) \subset X$ of $X \to C$ such that $\mathcal{O}_D(D) \simeq \mathcal{O}_D$ and $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is a nef line bundle (cf. [H2, V Proposition 2.8]). It is not difficult to see that $H^1(X, K_X \otimes \mathcal{O}_X(2D)) \to H^1(X, K_X \otimes \mathcal{O}_X(3D))$ is a zero map, $H^1(X, K_X \otimes \mathcal{O}_X(2D)) \simeq \mathbb{C}$, and $H^1(X, K_X \otimes \mathcal{O}_X(3D)) \simeq \mathbb{C}$. We note that $K_X \sim \mathcal{O}_X(-2D)$. Therefore, $\mathcal{O}_X(D)$ has no smooth hermitian metrics with semi-positive curvatures by Enoki's injectivity theorem (see [E, Theorem 0.2]). Note that $\kappa(X, \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)) = 0$ and $\nu(X, \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)) = 1$. We also note that Kollár's injectivity theorem implies nothing since $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is not semi-ample.

The next one is an example of nef and big line bundles that have no smooth hermitian metrics with semi-positive curvatures. I learned the following construction from Dano Kim.

Example 4.2. We use the same notation as in Example 4.1. Let $P \in C$ be a closed point. We put $\mathcal{F} := \mathcal{E} \oplus \mathcal{O}_C(P)$ and $Y := \mathbb{P}_C(\mathcal{F})$. Then it is easy to see that $\mathcal{L} := \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$ is nef and big (cf. [La, Example 6.1.23]). Since $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ has no smooth hermitian metrics with semi-positive curvatures, neither has $\mathcal{L}$. In this case, $H^i(Y, K_Y \otimes \mathcal{L}^k) = 0$ for $i > 0$ and any $k \geq 1$ by Kawamata-Viehweg vanishing theorem.

Let us recall some examples of semi-positive line bundles that are not semi-ample.

Example 4.3 (cf. [DEL, p.145])**.** Let C be a smooth projective curve with the genus $g(C) \geq 1$. Let $L \in \operatorname{Pic}^0(C)$ be non-torsion. We put $\mathcal{E} := \mathcal{O}_C \oplus L$ and $X := \mathbb{P}_C(\mathcal{E})$. Then $\mathcal{L} := \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is semi-positive, but not semi-ample. We note that $\kappa(X, \mathcal{L}) = 0$ since $H^0(X, \mathcal{L}^k) = H^0(C, S^k(\mathcal{E})) = H^0(C, \mathcal{O}_C) = \mathbb{C}$ for every $k \geq 0$, where $S^k(\mathcal{E})$ is the k-th symmetric product of $\mathcal{E}$. We can easily check that

$$K_X = \pi^*(K_C \otimes \det \mathcal{E}) \otimes \mathcal{L}^{-2} = \pi^*(K_C \otimes L) \otimes \mathcal{L}^{-2},$$

where $\pi : X \to C$ is the projection. Let m be an integer with $m \geq 2$. Then

$$H^i(X, K_X \otimes \mathcal{L}^m) = H^i(X, \pi^*(K_C \otimes L) \otimes \mathcal{L}^{m-2}) = \bigoplus_{k=1}^{m-1} H^i(C, K_C \otimes L^k).$$

Thus, $h^0(X, K_X \otimes \mathcal{L}^m) = (m-1)(g-1)$, $h^1(X, K_X \otimes \mathcal{L}^m) = 0$, and $h^2(X, K_X \otimes \mathcal{L}^m) = 0$ for $m \geq 2$. So, we obtain no interesting results from injectivity theorems. Note that $H^2(X, K_X \otimes \mathcal{L}^m) = 0$ for $m \geq 1$ also follows from Kawamata-Viehweg vanishing theorem since $\nu(X, \mathcal{L}) = 1$ and $\dim X = 2$.

Example 4.4 (Cutkosky)**.** We use the same notation as in Example 4.3. Let $P \in C$ be a closed point. We put $\mathcal{F} := \mathcal{O}_C(P) \oplus L$ and $Y := \mathbb{P}_C(\mathcal{F})$. Then it is easy to see that $\mathcal{M} := \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$ is big and semi-positive, but not semi-ample. We note that $\bigoplus_{m \geq 0} H^0(Y, \mathcal{M}^{\otimes m})$ is not finitely generated.

The following example shows the difference between Enoki's injectivity theorem and Kollár's one.

Example 4.5. Let C be a smooth projective curve with the genus $g(C) = g \geq 1$. Let $L \in \operatorname{Pic}^0(C)$ be non-torsion. We put $\mathcal{E} := \mathcal{O}_C \oplus$

$L \oplus L^{-1}$, $X := \mathbb{P}_C(\mathcal{E})$, and $\mathcal{L} := \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. It is obvious that $\mathcal{E}$ has a smooth hermitian metric whose curvature is Nakano semi-positive. Thus, $\mathcal{L}$ is semi-positive since $\mathcal{L}$ is a quotient line bundle of $\pi^*\mathcal{E}$, where $\pi : X \to C$ is the projection. In particular, $\mathcal{L}$ is nef. On the other hand, it is not difficult to see that $\mathcal{L}$ is not semi-ample. We have

$$K_X = \pi^*(K_C \otimes \det \mathcal{E}) \otimes \mathcal{L}^{-3} = \pi^* K_C \otimes \mathcal{L}^{-3}.$$

We can easily check that

$$S^m(\mathcal{E}) = \bigoplus_{\substack{0 \leq a+b \leq m \\ a,b \geq 0}} L^{a-b}.$$

Note that the rank of $S^m(\mathcal{E})$ is $\frac{1}{2}(m+2)(m+1)$. Let m be an integer with $m \geq 3$. Then it is easy to see that

$$H^i(X, K_X \otimes \mathcal{L}^m) = H^i(X, \pi^* K_C \otimes \mathcal{L}^{m-3}) = H^i(C, K_C \otimes S^{m-3}(\mathcal{E})).$$

for all i. We need the following obvious lemma.

Lemma 4.6. *We have $h^0(C, K_C) = g$ and $h^1(C, K_C) = 1$. Moreover, $h^0(C, K_C \otimes L^k) = g - 1$ and $h^1(C, K_C \otimes L^k) = 0$ for $k \neq 0$.*

Therefore, we obtain $H^3(X, K_X \otimes \mathcal{L}^m) = H^2(X, K_X \otimes \mathcal{L}^m) = 0$, $h^1(X, K_X \otimes \mathcal{L}^m) = \llcorner \frac{m-3}{2} \lrcorner + 1 = \llcorner \frac{m-1}{2} \lrcorner$, and

$$\begin{aligned} h^0(X, K_X \otimes \mathcal{L}^m) &= g \llcorner \frac{m-1}{2} \lrcorner + (g-1) \left(\frac{(m+2)(m+1)}{2} - \llcorner \frac{m-1}{2} \lrcorner \right) \\ &= \llcorner \frac{m-1}{2} \lrcorner + (g-1) \frac{(m+2)(m+1)}{2}. \end{aligned}$$

On the other hand, $h^0(X, \mathcal{L}^k) = h^0(C, S^k(\mathcal{E})) = \llcorner \frac{k}{2} \lrcorner + 1$ for $k \geq 0$. Let $s \in |\mathcal{L}^k|$ be a non-zero holomorphic section of $\mathcal{L}^k$ for $k \geq 0$. Then

$$\times s : H^1(X, K_X \otimes \mathcal{L}^m) \longrightarrow H^1(X, K_X \otimes \mathcal{L}^{m+k})$$

is injective by Enoki's injectivity theorem. Note that $h^1(X, K_X \otimes \mathcal{L}^m) = \llcorner \frac{m-1}{2} \lrcorner$ and $h^1(X, K_X \otimes \mathcal{L}^{m+k}) = \llcorner \frac{m+k-1}{2} \lrcorner$. We have $\kappa(X, \mathcal{L}) = 1$ by the above calculation. Since $\mathcal{L}^2 \cdot F = 1$, where F is a fiber of $\pi : X \to C$, $\nu(X, \mathcal{L}) = 2$. Thus, the nef line bundle $\mathcal{L}$ is not abundant. So, I think that there are no algebraic proofs for the above injectivity theorem. Note that $H^3(X, K_X \otimes \mathcal{L}^m) = H^2(X, K_X \otimes \mathcal{L}^m) = 0$ for $m \geq 1$ follows from Kawamata-Viehweg vanishing theorem since $\nu(X, \mathcal{L}) = 2$ and $\dim X = 3$.

The last two examples are famous ones due to Mumford and Ramanujam.

Example 4.7 (Mumford)**.** Let us recall the construction of Mumford's example (see [H1, Example 10.6]). We use the same notation as in [H1, Example 10.6]. Let C be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. Then there exists a stable vector bundle E of rank two and $\deg E = 0$ such that its symmetric powers $S^m(E)$ are stable for all $m \geq 1$. We consider the ruled surface $X := \mathbb{P}_C(E)$. Let D be the divisor corresponding to $\mathcal{O}_X(1)$. Since E is a unitary flat vector bundle, $\mathcal{L} := \mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}_C(E)}(1)$ is semi-positive by $\pi^*E \to \mathcal{L} \to 0$, where $\pi : X \to C$ is the projection. We know that $H^0(X, \mathcal{L}^m) = H^0(C, S^m(E)) = 0$ since $S^m(E)$ is stable and $c_1(S^m(E)) = 0$ for any $m \geq 1$. Thus, $\kappa(X, \mathcal{L}) = -\infty$. On the other hand, $\mathcal{L} \cdot \mathcal{L} = 0$ and $\mathcal{L} \cdot C > 0$ for any curve C on X. Then $\nu(X, \mathcal{L}) = 1$.

Example 4.8 (Ramanujam)**.** Let us recall the construction of Ramanujam's example (see [H1, Example 10.8]). We use the same notation as in [H1, Example 10.8]. Let X be the ruled surface obtained in Example 4.7. We assume that D is the divisor given in [H1, Example 10.6] (see Example 4.7 above). Let H be an effective ample divisor on X We define $\overline{X} := \mathbb{P}_X(\mathcal{O}_X(D-H) \oplus \mathcal{O}_X)$, and let $\pi : \overline{X} \to X$ be the projection. Let X_0 be the section of π corresponding to $\mathcal{O}_X(D-H) \oplus \mathcal{O}_X \to \mathcal{O}_X(D-H) \to 0$ and $\overline{D} := X_0 + \pi^*H$. We put $\mathcal{M} := \mathcal{O}_{\overline{X}}(\overline{D})$. We write $\mathcal{O}_{\overline{X}}(1) = \mathcal{O}_{\mathbb{P}_X(\mathcal{O}_X(D-H)\oplus\mathcal{O}_X)}(1)$. Then $\mathcal{O}_{\overline{X}}(1) \simeq \mathcal{O}_{\overline{X}}(X_0)$. Therefore, $\mathcal{M} \simeq \mathcal{O}_{\overline{X}}(1) \otimes \pi^*\mathcal{O}_X(H)$ and $\pi^*(\mathcal{O}_X(D) \oplus \mathcal{O}_X(H)) \to \mathcal{M} \to 0$. Thus, it is easy to see that $\mathcal{M}$ is semi-positive, nef and big. By the construction, $\mathcal{M} \cdot C > 0$ for any curve C on $\overline{X}$. However, $\mathcal{M}$ is not semi-ample since $\mathcal{M}|_{X_0} \simeq \mathcal{O}_{X_0}(D)$ does not have sections on X_0. In particular, $\bigoplus_{m\geq 0} H^0(\overline{X}, \mathcal{O}_{\overline{X}}(m\overline{D}))$ is not finitely generated.

We close this section with the following question.

Question 4.9. Let X be a smooth projective variety and $\mathcal{L}$ a line bundle on X. Assume that $\mathcal{L} \cdot C > 0$ for any curve C on X. Then, is $\mathcal{L}$ semi-positive?

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